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CONTRIBUTIONS TO CONSUMER THEORY COHERENCE AND PARTIAL COHERENCE

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Abstract

The behavioural theory of consumer is revised using the mathematical tools provides by the theory of external differential forms. After a brief outlook of the Pareto's classical theory a generalisation, using tools normally employed in mathematics but not in economics, is presented. Definition of partial coherence is then introduced to generalise the concept of coherence. Finally, some suggestions for applying the proposed procedures to other topics of economic theory are introduced.

1. INTRODUCTION

The theory that does its best to describe and explain the behaviour of economic agents called "consumers" is essentially classical: by way of an example, suffice it to recall how Vilfredo Pareto applied to the consumer problem the well-known procedures that Lagrange had worked out in his inquiry into the conditioned maximums or minimums of the functions of several variables.

Our aim in this paper is to return to the subject using the mathematical tools provided by the theory of external differential forms.

To this end we will first take a quick look at the classical theory, and then present the generalizations obtained by using tools that are classical in mathematics but totally different to those normally employed in economic theory. This application will allow us to introduce the concept of partial coherence with which we hope to be able to generalize the concept of coherence. In fact this latter concept could be used to describe the behaviour of the consumer in the cases embraced by the classical theory (which we shall call Paretian) but does not lend itself to generalization within the sphere of this theory. However, we believe that by using mathematical tools that are more powerful than those used in Paretian theory it should be possible to widen the concept of coherence, thereby furthering our analysis of consumers behaviour by means of mathematical tools.

We should also like to observe that in our view the procedures we are introducing could also be usefully adopted in other chapters of economic theory, thus providing a wider and deeper understanding of the problems addressed.

2. THE CLASSICAL THEORY

In this paper we will make methodical use of geometrical language because it is convenient and stimulating for the purposes of exposition.

This convention is clearly not harmful to generality, which means that it is highly unlikely to create interpretative difficulties with tegard to the problems of economic theory that we shall be addressing here.

Let X stand for a real Euclidean space with n dimensions (with $n \geq 3$); the hypothesis that the space is Euclidean will fa cilitate the expression of certain equations whose meanings will remain, as we have said, exclusively economic.

Let us adopt the following conventions:

$$x = [x_1, x_2, \ldots, x_n]. \tag{2.1}$$

to indicate that the coordinates of x are the (real) numbers $x_1, x_2, \ldots, x_n$.

We will use the symbol X' to indicate the subset of the points of X whose coordinates are all positive; thus we can suppose that:

$$X' = \{x \in X \mid x_i > 0,\ 1 \leq i \leq n\}. \tag{2.2}$$

If A stands for an open set, simply connected and limited, contained within X', then:

$$A \subset X'. \tag{2.3}$$

The coordinates x_i ($1 \leq i \leq n$) stand for the quantities of goods that a given consumer can purchase.

The classical or Paretian theory assumes the existence of an index function of utility (or ophelimity, to use Pareto's terminology); let u stand for this function, and let us suppose that it has real values and is defined and continuous in the open set A. We will thus obtain:

$$u: A \rightarrow \mathrm{IR} \;;\; x \mapsto u(x). \tag{2.4}$$

It is clearly reasonable to suppose that the function u(x) is such that it could establish a correspondence between the set A and the real straight line IR, such that the total arrangement within IR makes it possible to set up a total arrangement within the open set A as well. Thus, given two points x, y of A, it can be said that the consumer prefers the goods possession situation indicated by the coordinates of x to that indicated by the coordinates of y if and only if :

$$u(x) > u(y). \tag{2.5}$$

"Indifference variety" is the name given to a set of points of A in which

$$u(x) = \text{constant.} \tag{2.6}$$

Without pausing to discuss the meaning of the numerical value of the function u at a point x of A, suffice it to recall that when there is a function of ophelimity that performs the functions attributed here to u, then any other function:

$$F[u(x)], \tag{2.7}$$

where F is a continuous, monotonous in the strict sense and growing function, can be used as the utility index (ophelimity) function.
The correspondence established by the function u between the open set A and the real straight line IR thus transforms the relation of preference between the two situations indicated by the two points x and y in the comparison of real values taken from the function u in the same points; and the existence of a total arrangement on the real straight line allows us to conclude that the relation of preference thus established possesses the following properties:

a) two situations, corresponding to two points x and y of A, can always be compared with each other. In other words, the consumer (who translates his preferences with the values of the function u) is always able to determine whether or not the situation indicated by the coordinates of point x compared with that of the coordinates of y is indifferent for him, or whether he prefers one situation to the other;

b) if a situation corresponding to the coordinates of x is preferred to the one corresponding to the coordinates of y, and if this in its turn is preferred to the one that corresponds to the coordinates of a point z, then the situation corresponding to the coordinates of x is preferred to the one corresponding to z. In mathematical terms it can be said that in this case the preference relationship thus established by the utility function possesses the transitive property.

We agree that the consumer who establishes his preferences in the manner thus described is "globally coherent" in the open set A.

3. THE UTILITY FUNCTION

In the mathematical treatment it is generally supposed that the utility function u possesses certain properties by means of which certain problems and their solutions can be formulated using the tools of mathematical analysis. The properties that we assume to be valid for the function u are the following:

a) in the open set A, u possesses derivatives of the first and second order, and these latter are continuous throughout the open set A. In the language of mathematical analysis, this would be expressed by saying that in the open set A the function u is of class 2 derivability at least.

For the sake of brevity, we shall be adopt the following notations for the primary and secondary partial derivatives of function u:

$$u_i = \frac{\partial u}{\partial x_i} \tag{3.1}$$

and also:

$$u_{ik} = \frac{\partial^2 u}{\partial x_i \partial x_k}. \tag{3.2}$$

The classical theorems of mathematical analysis allow us to conclude that the hypotheses assumed valid for the function u can guarantee that for the secondary partial derivates of u the following equations are valid:

$$u_{ik} = u_{ki} \quad (1 \leq i, k\ n). \tag{3.3}$$

b) In the open set A the hypersurfaces of indifference, expressed by the equation:

$$u = \text{cost.} \tag{3.4}$$

are convex in the strict sense. Therefore, considering two points x, y of A, and considering two real numbers a and b that satisfy the following conditions:

$$a > 0,\ b > 0,\ a + b = 1, \tag{3.5}$$

if we assume:

$$z = a\,x + b\,y, \tag{3.6}$$

then we will always obtain:

$$u(z) > u(x)\ ,\ u(z) > u(y). \tag{3.7}$$

4. THE LAGRANGE-PARETO EQUATIONS

The hypotheses stated in the previous paragraph for the function u clearly allow us to use the language of mathematical analysis to translate the consumer's problem; indeed, to solve this problem using the tools worked out by Lagrange for research into the extremal conditioned values of the functions of several variables.

To this end we assume the existence of a vector p that we shall call the price vector, whose components are the prices of the goods available on rhe market for purchase on the part of the consumer:

$$p = [p_1, p_2, \ldots, p_n]. \tag{4.1}$$

Let the real, non negative number R stand for the overall expenditure of the consumer, that is the quantity of money he devotes to the purchase of the available goods. We will thus obtain the basic equation:

$$R = \sum_i p_i x_i \ , \tag{4.2}$$

that expresses the overall expenditure that the consumer devotes to the purchase of goods.

From this equation, often called the "budget equation", we can obtain the following one:

$$\frac{\partial R}{\partial p_i} = x_i \tag{4.3}$$

to which we shall be referring later.

Let us now assume that the consumer's behaviour tends to seek the point x that corresponds to the maximum value of u when the sum R is fixed, or to minimize the sum R when the value of the function u is fixed. As we shall be showing, this proposition expresses the basic hypothesis of the consumer's behaviour, provided that there is a utility function.

If the above-mentioned hypothesis is translated by means of mathematical analysis, we obtain what we shall call the Lagrange system of equations:

$$\begin{cases} u_i = & h p_i \\ R = & \sum_i p_i x_i \end{cases} \tag{4.4}$$

where h is a real number that is usually known as the "Lagrange multiplicator". By eliminating h from the equation of the first line at (4.4) we obtain the following system of equations:

$$p_i u_k - p_k u_i = 0 \qquad (1 \le i, k \le n). \tag{4.5}$$

These amount to n(n - 1)/2 in number, although obviously only (n - 1) of them are linearly independent.

With the proposed hypotheses, the system based on the (4.5) equations and the second (4.4) one allows us to consider the p_j as functions of x_k or these latter as functions of the p_j, in the open set A, when the stated hypotheses are valid and when the consumer regulates his behaviour in accordance with the basic hypothesis stated above.

In particular it is thus possible to consider certain n functions:

$$x = f_i(p|R) \tag{4.6}$$

that express the quantities of goods purchased by the consumer in the stated hypothetical conditions and in relation to the price vector p and the overall expenditure R.

The (4.6) functions are known as "consumer demand functions".

5. THE INDIFFERENCE FACET

Using conventional geometrical language as mentioned in paragraph 1, the content of the equations formulated at (4.4) of the preceding paragraph can be expressed by saying that for each point x of the open set A there is an "indifference facet" for the consumer. This facet can be represented as follows. Let us assume that the linear polynomial in the differentials dx_i is:

$$\pi = \sum_i p_i dx_i \, . \tag{5.1}$$

This linear polynomial is also known as the "Pfaff form", or as "Pfaffian". Using the evocative language of differentials it can be said that the value of the Pfaffian π represents the (infinitesimal) expenditure made by the consumer to change his situation, moving within a constant price context from point x of A to a point whose coordinates have grown in algebraic terms (that is, with possible negative growth) of the quantities dx_i .

The equation:

$$\pi = \sum_i p_i dx_i = 0 \tag{5.2}$$

translates the condition whereby the infinitesimal increment is tangent to the hypersurface of indifference, and thus takes place with constant utility.

The (5.2) equation is called a Pfaff's equation. We will be dealing with it later in this paper, using the methods and approach of a classical theory that handles individual equations of this type and systems of equations of the same sort.

For the moment we would like to make one or two observations that explain the aim of this paper and the introductory words in paragraph 1.

We should first of all point out that the Pfaff's equation (5.2) derives from the stated hypotheses, that is from the existence of a consumer utility function and from the consumer's behaviour in accordance with the basic hypothesis. In other words, in these conditions demand functions can be taken into consideration, and the consumer's behaviour is globally coherent. This in its turn means that the consumer can express peference (or indifference) regarding two situations corresponding to two points x and y of A, even when these are distant from each other. However, the (5.2) equation expresses consumer's behaviour that only concerns situations that are very close and presupposes the existence of demand functions alone and not of global utility functions. In other words, the validity of (5.2) can be derived from the classical hypotheses, but this does not ensure the validity of the classical hypotheses. To achieve this, further conditions need to be verified; and these, along with the (5.2) equation, define the globally coherent behaviour of the consumer. If this is not so, a more limited concept of coherence can be defined, as we shall see.

We should also like to point out that, from the point of view of the application of the concepts that we are currently explaining, it would appear to be easier and more effective to work out the observation procedures aimed at verifying equation (5.2) than to construct procedures for demonstrating the existence of a utility index function in the open set A.

6. THE SLUTSKY EQUATIONS

As we have seen, classical Paretian theory provides us with a description of consumer behaviour with the (4.4) system of equations. What is known as "Slutsky's equations" derive from these equations using derivation procedures. The Slutsky's equations are as follows:

$$\frac{\partial p_i}{\partial x_k} + x_k \frac{\partial x_i}{\partial R} = \frac{\partial p_k}{\partial x_i} + x_i \frac{\partial x_k}{\partial R}. \tag{6.1}$$

For their demonstration see manuals dealing with mathematical economy such as the one by Manara-Nicola (1970).

What interests us here is to obtain these equations via the Pfaff's equations mentioned in paragraph 5. To this end we can calculate the external differential of the Pfaff's form π defined by equation (5.1) of paragraph 5. In fact from (4.6), bearing in mind the equation (4.3) in the same paragraph, and assuming:

$$s_{ik} = \frac{\partial p_i}{\partial x_k} + x_k \frac{\partial x_i}{\partial R} \tag{6.2}$$

we obtain:

$$d\pi = \sum_{i<k} (s_{ik} - s_{ki}) dp_k \wedge dp_i \;. \tag{6.3}$$

Thus the subsistence of the Slutsky equations implies that the Pfaffian differential form p is closed, such that:

$$d\pi = 0. \tag{6.4}$$

Vice-versa, the theory of external differential forms ensures that, in the hypotheses formulated for the open set A, the subsistence of (6.4) is also a sufficient condition for ensuring that the Pfaffian equation:

$$d\pi = 0. \tag{6.5}$$

is completely integrable, in other words for ensuring that there is a sheaf of surfaces

$$u(x) = \text{constant} \tag{6.6}$$

such that for each point of A

$$du = \pi . \tag{6.7}$$

The theory mentioned also shows that the sufficient condition for the complete integrability of the Pfaffian equation (6.5) can be expressed in a more general form, imposing the existence of a differential form θ such that:

$$d\pi = \pi \wedge \theta . \tag{6.8}$$

Thus, within this framework is can be said that the subsistence of the Slutsky equations is both a necessary and a sufficient condition for ensuring the global coherence of the consumer's behaviour within the open set A.

7. PARTIAL COHERENCE

We have seen that the theory of external differential forms gives an interesting meaning to the classical Slutsky equations in connection with the problem of consumer coherence. At this point we can take a closer look at the meaning of the more general situation that arises when we know the consumer's demand functions and we can thus construct the Pfaffian form p, even without satisfying the Slutsky equations.

Obviously in this case there is no utility function that can act as a choice criterion that the consumer can use to compare two situations and make coherent decisions. However, here again certain characteristics of the consumer's behaviour can be taken into consideration, thereby introducing a somewhat milder concept of coherence.

To this end let us point out that even when (6.8) does not apply, it is possible to define the natural number p that satisfies the limitation:

$$2p \leq n - 1 \tag{7.1}$$

and there are 2p functions:

$$z_1, z_2, \ldots, z_p, v_1, \ldots, v_p \tag{7.2}$$

such that the Pfaff's polynomial π can be expressed in the following form:

$$p = du + z_1\, dv_1 + z_2\, dv_2 + \ldots + z_p\, dv_p\,. \tag{7.3}$$

It thus follows that the Pfaffian equation $\pi = 0$ is verified on the n -p-1 dimensional variety defined by the equations:

$$u = c \text{ (constant)}; \; v_i = c_i \; (1 \leq i \leq p). \tag{7.4}$$

This variety can thus be considered as the immediate generalization of the indifference variety of the classical Paretian theory and could thus be called limited or partial indifference variety.

Moreover it should be observed that the (7.4) equations constitute a system of p+1 equations in finite terms that link the coordinates of a point x belonging to the open set A. Thus there are p+1 varieties of dimension n-p, each one of which is represented by p equations selected from the (7.4) ones. One of these, for example, is defined by the equations:

$$v_i = c_i \;\; (1 \leq i \leq p); \tag{7.5}$$

Only function u varies on this variety, which means that its values can be used by the consumer as a criterion for comparing two situations. It is thus reasonable to talk about weak or partial coherence on the part of the consumer, who decides to keep on a variety defined by equations (7.5) and regulates his choices on the value adopted by function u (at points in the variety considered, of course). Similar considerations obviously also apply to each of the other varieties defined by the p equations selected from the (7.5) ones.

By referring to the language of the theory of contact transformarions, it can thus be said that if the Pfaff's equation (6.5) is completely integrable, then the indifference facets can be organized such that they are all tangents of the hypersurfaces of a (6.6) type sheaf. If on the other hand the Pfaff's equation is not completely integrable, then the facets can be organized so that they are tangents to certain varieties smaller in size than (n - 1).

8. OBSERVATIONS

At this point we should like to make a few brief observations concerning the significance and eventual effects of the theoretical developments hitherto described.

Firstly we should point out that we have been referring to the classical problem of the consumer solely in order to clarify our ideas and avoid the excessively abstract nature of general treatments that could appear far removed from any possible application to real economic problems. In actual fact it would be relatively easy to extend the concepts and developments outlined here to problems of a much more general nature; indeed, to many other economic questions.

The first example to focus on here derives from the theory of production and the mathematical schematization of the related problems using methods pertaining to profit optimization under given constraints. However we are also convinced that within the same framework it should be possible to deal with numerous other problems concerning economic systems if the problems themselves are formulated as a quest for the extreme conditioned values of certain functions.

In the second place we believe that our framework should help overcome the distance that appears to separate the various approaches to particular economic problems. In fact certain mathematical approaches which seem very elegant to the mathematician are criticized as being excessively abstract because they assume the economic agent's behaviour to be globally coherent, which it rarely is, or because they presuppose the possibility of making comparisons between very different situations. However, we believe that mathematical tools can help schematize and deal with situations in which the agents only display a sort of limited (or local) coherence. This in its turn may be seen as a step in the direction of theoretical tools that actually fit the object of the inquiry better.

Appendix. A reminder of the external differential forms

1. The theory of external differential forms is not often used in economic analysis, so we shall now proceed to mention some of its salient features. As we have already said, with the geometrical language we have chosen to use we obtain agile and concise expressions without detracting from the generality. The notation used here has absolutely no relation to the meanings given to the symbols in the previous paragraphs.

Alongside the n-sized Euclidean space in which a point x has the following (real) coordinates:

$$x_1, x_2, \dots, x_n, \tag{A.1}$$

let us consider the vector space of the differentials of the variables.

Let us assume:

$$dx = [dx_1, dx_2, \dots, dx_n]. \tag{A.2}$$

On this vector space of the differentials let us then construct the n(n-1)/2-sized space whose generators are the orderly pairs of elements dx_i and dx_k. We shall use the symbol:

$$dx_i \wedge dx_k \tag{A.3}$$

to indicate an element of the second vector space that we believe derives from an "alternate product" operation applied to the two differentials dx_k and dx_i. This product is indicated by the presence of the symbol " $\wedge$ " between the two differentials and is held to yield to the syntax expressed by the following formal rules:

$$\begin{cases} dx_i \wedge dx_k + dx_k \wedge dx_i = 0 \quad \text{and therefore} \\ dx_i \wedge dx_i = 0 \\ dx_i \wedge \left[a \cdot dx_k + b \cdot dx_j\right] = a \cdot dx_i \wedge dx_k + b \cdot dx_i \wedge dx_j \\ \text{with a and b real numbers} \end{cases} \tag{A.4}$$

2. Let A be an open set simply connected in the space, and $a(x)$ stand for a vector whose components are functions of the point $\mathbf{x}$ of A:

$$\mathbf{a}(\mathbf{x}) = [a_1(\mathbf{x}), a_2(\mathbf{x}), \dots, a_n(\mathbf{x})]. \tag{A.5}$$

We shall assume that the functions that make up the vector a have at least derivatives of first and second order in A, and that the latter are continuous.

The first degree polynomial in the dx_i :

$$a = \sum_i a_i dx_i \tag{A.6}$$

is also called "linear differential form" or Pfaff's form.

In the hypotheses thus stated, it is also possible to construct the second degree polynomial (in the dx_i variables) expressed by the formula:

$$d\alpha = \sum_k da_k \wedge dx_k \ , \tag{A.7}$$

where the symbol da_k stands for the total differential of the function a_k, that is:

$$da_k = \sum_j \frac{\partial a_k}{\partial x_j} \cdot dx_j \tag{A.8}$$

The polynomial $d\alpha$ expressed by (A.7) is called the "external differential" of the Pfaffian form α; bearing in mind the syntactical rules of the external product, it can be expressed in the following form:

$$da = \sum_{j<k} \left(\frac{\partial a_k}{\partial x_j} - \frac{\partial a_j}{\partial x_k} \right) dx_j \wedge dx_k \, . \tag{A.9}$$

3. Let us consider the particular case in which the Pfaff's form derived from (A.6) in the previous paragraph expresses the total differential of a function u; in other words be:

$$\alpha_i = \frac{\partial u}{\partial x_i} \tag{A.10}$$

and thus:

$$\alpha = du. \tag{A.11}$$

In this case we obtain

$$d\alpha = \sum_{j<k}\left(\frac{\partial^2 u}{\partial x_k \partial x_j} - \frac{\partial^2 u}{\partial x_j \partial x_k}\right)dx_j \wedge dx_k \tag{A.12}$$

such that, assuming the function u to yield the aforesaid hypothesis in the open set A:

$$d\alpha = 0. \tag{A.13}$$

A Pfaff's differential form for which (A.13) is valid is called closed; if it is the total differential of a function u, that is if (A.11) is true, then it is called exact. The brief calculation performed above shows that every exact form is closed.

4. The equation:

$$\alpha = 0 \tag{A.14}$$

is called a "Pfaff's equation". The equation (A.14) is said to be "completely integrable" if there are two function: u(x) and F(x) such that:

$$F\alpha = du. \tag{A.15}$$

The function F is called the "integrating factor of the form α", and each variety represented by an equation of the type:

$$u(x) = c \text{ (constant)} \tag{A.16}$$

(or represented in an equivalent form) is called the "solution" or "integral" of the equation (A.14).

Obviously if $F = 1$ then the form α is exact, according to the terminology introduced in paragraph 3.

The necessary and sufficient condition for equation (A.14) to be completely integrable thus proves to be the existence of another form

$$\beta = \sum_k b_k(x)dx_k \tag{A.17}$$

such that:

$$d\alpha = \alpha \wedge \beta. \tag{A.18}$$

If this condition is not met, then there is a natural number that meets the requirements of the relation:

$$2p \leq n - 1 \tag{A.19}$$

and there are 2p functions:

$$z_1, z_2, \ldots, z_p, v_1, v_2, \ldots, v_p \tag{A.20}$$

such that the Pfaff differential form α can be represented as follows:

$$\alpha = du + \sum_{i=1}^{p} z_i dv_i \; . \tag{A.21}$$

In this case the $n - p - 1$ dimensional varieties represented by the system of equations:

$$u = c \text{ (constant)}; \; v_i = c_i \text{ (constants)} \tag{A.22}$$

can be called "solutions" or "integrals" of the equation (A.14).

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